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First integrals of non-holonomic systems and their generators

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Abstract. We discuss various aspects of mechanical systems with general (nonlinear) non-holonomic constraints from the perspective of presymplectic geometry. We begin by introducing a 2-form on the evolution space of a system having the property, among others, of modelling the unconstrained dynamics. Using this 2-form we then characterize a unique second-order dynamics on the constraint submanifold through a simple geometrical implementation of Chetaev's concept of virtual work. We also give necessary and sufficient conditions in order for the reduced dynamics to admit a non-holonomic Lagrangian formulation. Finally, we study the structure of a set of vector fields on the constraint submanifold which generates all first integrals of a constrained system. The relationships with a previously proposed set of vector fields in non-conservative holonomic mechanics and with known generalizations of Noether's theorem for non-holonomic systems are analysed.

1. Introduction

Despite the long history of non-holonomic mechanics, the establishment of a geometric context allowing one to produce practical links with corresponding problems in holonomic mechanics still requires much development, mostly concerning symmetries and first integrals. The main difficulty consists in the fact that, contrary to the holonomic case, a symmetry of a non-holonomic system does not yield in general a constant of the motion. On the other hand, in the last few years several papers (see, e.g., [16] and references therein) have been concerned with the problem of constructing first integrals of non-holonomic systems by means of transformations defined in one way or another. The purpose of the present paper is to analyse, within the framework of presymplectic geometry [33], some aspects of the dynamics of non-holonomically constrained systems with a focus on first integrals and their generators. The analysis is carried on in a frame-independent language, using the standard tools of jet-bundle theory. Recent progress in this direction have been made by, among others, Bates and Śniatycki [2], Koiller [13], Giachetta [11], Cushman *et al* [8], Bloch *et al* [4], de León *et al* [14, 15], Massa and Pagani [20–22], Marle [19], Sarlet *et al* [28, 29], Śniatycki [32], Morando and Vignolo [23], Saunders *et al* [30], Giachetta *et al* [12], Cortés and de León [6].

An outline of the main results of this paper is as follows. Let $\tau : E \to \mathbb{R}$ be a fibre bundle and $\pi : J^1\tau \to E$ its first-order jet bundle. As is well known [7] (see also [33], p 132), given a second-order differential equation (SODE) field ξ on $J^1\tau$, one can construct a 2-form Ω of maximal rank on the same manifold such that ξ is a characteristic vector field. The construction of this 2-form is based on an Ehresmann connection Γ on $J^1\tau \to E$ such that the given SODE field ξ is contained in its horizontal distribution. Let $i : C \hookrightarrow J^1\tau$ be an embedded submanifold fibred over E representing some (in general, nonlinear) kinetic

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constraints imposed on a mechanical system. Although there is no general agreement in the literature about the mathematical scheme to deal with nonlinear non-holonomic constraints, the most widely spread model (and that used in this paper) is based on the so-called 'Chetaev rule', whose geometrical significance is clarified by the construction of the Cheatev bundle [22]. Building on the d'Alembert principle, which is assumed to remain valid in the general case, we show that the cotangent bundle T^*C of the constraint submanifold decomposes in a four-way split. This splitting determines a SODE field $\widehat{\xi}$ on C whose content comprises, by construction, both Chetaev's rule and d'Alembert's principle, so it characterizes the dynamics of a non-holonomic system. This is much the same way as in the holonomic case in which the tangent bundle $TJ^1\tau$ of the evolution space decomposes into a direct sum of three vector bundles, one of which is spanned by the SODE field ξ [7]. An immediate corollary of our theorems is the characterization given in [30] (see also [28, 29]), for Lagrangian systems, of the reduced dynamics on the constraint submanifold as the unique SODE field in the one-dimensional kernel of a certain 2-form (derived from the Cartan 2-form).

Let $\widehat{\Omega}$ be the pull-back of the 2-form Ω on C. In [23], the authors introduce the notion of a non-holonomic Lagrangian for a SODE field on the constraint submanifold. This is a pair consisting of a function on C and a Chetaev form (i.e. a section of the Chetaev bundle) by means of which one can construct the analogue of the Cartan 1-form. For each SODE field on C obtained in the way explained above, we give a necessary and sufficient condition in order for it to be derivable from a non-holonomic Lagrangian. Essentially, this can be stated as $\mathrm{d}\widehat{\Omega} \in \mathrm{d}\mathcal{I}$, where $\mathcal{I}$ denotes the ideal generated by the module of Chetaev forms. It turns out that this condition also plays an important role in the subsequent discussion on vector fields generating first integrals.

Motivated by [5], in the second part of the paper we consider a set $\mathcal{V}$ of vector fields on the constraint submanifold which directly generate first integrals of the constrained dynamics. More precisely, we show that there is a one-to-one correspondence between equivalence classes of vector fields in $\mathcal{V}$, where two vector fields are identified if they differ by a multiple of $\widehat{\xi}$, and equivalence classes of first integrals, where two first integrals are equivalent if their differentials differ by a Chetaev form. We note that $\mathcal{V}$ contains (in fact is much larger than) the 'special class of Noether vector fields' considered in [23]. The set of vector fields studied in the present paper is also related to some extent with several works (see, e.g., [16]) aimed at a generalization of Noether's theorem for non-conservative non-holonomic systems. In particular, $\mathcal{V}$ can be regarded as the geometrical counterpart of the set of infinitesimal transformations which are the solutions of the so-called generalized Killing equations.

Many properties of the corresponding set of vector fields in the non-constrained case [5] also hold for $\mathcal{V}$. The vector fields in $\mathcal{V}$ are not in general dynamical symmetries. We show that in order for this to be the case it is enough that the constraint submanifold is integrable and $\mathrm{d}\widehat{\Omega} \in \mathrm{d}\mathcal{I}$. Actually, under these conditions we recover, on every leaf of C, the characterization of Noether symmetry for holonomic systems. Another circumstance in which every vector field in $\mathcal{V}$ is a dynamical symmetry occurs when the SODE field $\widehat{\xi}$ is a symmetry, i.e. it preserves both the 2-form $\widehat{\Omega}$ (up to an element of $\mathcal{I}$) and the Chetaev bundle.

The scheme of the paper is as follows. In section 2 we recall some basic features of jet spaces which are needed for a frame-independent description of a mechanical system. Moreover, we show how to characterize the dynamics using a 2-form of maximal rank on the evolution space defined by means of an Ehresmann connection. In section 3 a basic geometric set-up is laid out that enables one to model velocity-dependent constraints. Here we discuss the way in which the (Chetaev) definition of virtual displacement and the d'Alembert principle lead to the determination of a unique SODE field on the constraint submanifold. We also study the conditions under which such a SODE field is derivable from a non-holonomic Lagrangian.

In section 4 we study the relationship between first integrals on the constraint submanifold and their generators. Among others topics, the discussion includes the relations of these generators with dynamical symmetries and their role in well known results about a generalized version of Noether's theorem for non-conservative non-holonomic systems. The last section contains some illustrative examples.

Throughout the paper all objects are smooth (i.e. C^∞). All manifold are real, finite-dimensional, second-countable (hence paracompact) and connected. For convenience, we usually do not distinguish between a vector bundle and the set of its smooth sections. The only exception to this rule is the use of $\mathcal{D}(M)$ and $C^\infty(M)$ for the set of smooth vector fields and the set of smooth functions on a manifold M, respectively. The Lie derivative of a form α with respect to a vector field X is denoted by $L_X\alpha$, whereas the inner product of X and α is written as $Z\rfloor\alpha$. Finally, $X(h)$ denotes the Lie derivative of a function h with respect to X.

2. Preliminaries

Let E be the configuration spacetime manifold of a mechanical system, with the usual fibre bundle structure $\tau : E \to \mathbb{R}$ over the absolute time and standard fibre M. At this stage we only focus on the positional constraints of the system, the additional kinetic constraints are described in a subsequent step. Although E will be trivial, i.e. $E \cong \mathbb{R}\times M$, no one trivialization of it is to be preferred to any other. Consequently, by working in this way we shall ensure that all our formulae are tensorial with respect to time-dependent coordinate transformations. From the physical viewpoint, a trivialization of $E \to \mathbb{R}$ corresponds to a frame of reference.

Every section $\gamma : \mathbb{R} \to E$ represents a possible history of the system. Therefore, the first-order jet manifold $J^1\tau$ can be regarded as the totality of admissible kinetic states of the system. We shall now review some aspects of the geometry of $J^1\tau$. More details can be found in [10, 31].

2.1. Geometry of $J^1\tau$

From an algebraic viewpoint, we recall that $\pi : J^1\tau \to E$ is an affine bundle modelled on the vector bundle $V\tau \to E$ of vectors tangent to the fibres of τ, henceforth called the vertical bundle over E. By definition, both $V\tau$ and $J^1\tau$ may be identified with corresponding sub-bundles of the tangent bundle TE. Introducing coordinates (t, q^i) on E, $(t, q^i, \dot{q}^i)$ on $J^1\tau$ and $(t, q^i, \dot{t}, \dot{q}^i)$ on TE, $J^1\tau$ and $V\tau$ are locally defined by $\dot{t} = 1$ and 0, respectively. If $\dim E = n + 1$, then $\dim J^1\tau = 2n + 1$ and $\dim TE = 2n + 2$.

Any coordinate transformation $(t, q^i) \to (t', q'^i)$ on E, where $q'^i = q'^i(t, q^j)$ and $t' = t$, induces a transformation of velocities given by

$$\dot{q}'^i = \frac{\partial q'^i}{\partial t} + \frac{\partial q'^i}{\partial q^j}\dot{q}^j. \tag{1}$$

The affine structure of $J^1\tau \to E$ guarantees that the vector bundle $V\pi \to J^1\tau$ of vectors tangent to the fibres of π, hereafter called the vertical bundle over $J^1\tau$, is canonically isomorphic with the pull-back bundle $\pi^*V\tau$. By means of this identification we can lift vertical vectors from E to $J^1\tau$. In local coordinates this lift reads

$$X = X^i\frac{\partial}{\partial q^i} \quad \mapsto \quad X^v = X^i\frac{\partial}{\partial \dot{q}^i}. \tag{2}$$

The ordinary Euclidean structure of the physical 3-space induces a symmetric bilinear form g on the vertical bundle $V\tau$, henceforth called the fibre metric, which is a frame-independent attribute of $E \to \mathbb{R}$. The positivity condition $g(X, X) > 0$ holds for all vertical vectors $X \neq 0$. In local coordinates the fibre metric is represented by the matrix

$$g_{ij} = g\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right).$$

Note that, by means of the vertical lift, we can regard g as a bilinear form on the vector bundle $V\pi$. According to (2), locally we put

$$g\left(\frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{q}^j}\right) = g\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right). \tag{3}$$

We also recall that $J^1\tau$ admits a canonical linear endomorphism $J : TJ^1\tau \to TJ^1\tau$, henceforth called the vertical endomorphism, which generalizes the vertical endomorphism on a tangent bundle. Its coordinate expression is given by

$$J = \theta^i \otimes \frac{\partial}{\partial \dot{q}^i} \qquad \theta^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t. \tag{4}$$

A glance at (4) shows that the kernel of J is the $(n+1)$-dimensional distribution on $J^1\tau$ spanned by $J^2\tau$, the second-order jet manifold of $\tau : E \to \mathbb{R}$, whereas the image of J is $V\pi$. By duality, the vertical endomorphism gives rise to a morphism $J^* : T^*J^1\tau \to T^*J^1\tau$ which locally reads

$$J^* = \frac{\partial}{\partial \dot{q}^i} \otimes \theta^i.$$

Elementary considerations yield that the image of J^*, hereafter denoted by $\mathcal{C}(J^1\tau)$ and called the contact bundle over $J^1\tau$, coincides with the annihilator of the kernel of J. The sections of the contact bundle will be called contact 1-forms. A local basis of $\mathcal{C}(J^1\tau)$ is provided by $\theta^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t$.

An Ehresmann connection on the bundle $J^1\tau \to E$ is a vector sub-bundle of $TJ^1\tau$ which is complementary to the vertical sub-bundle $V\pi$. The corresponding distribution on $J^1\tau$ is called the horizontal distribution. Later on, we refer to an Ehresmann connection always in terms of the vertical projection $\Gamma : TJ^1\tau \to TJ^1\tau$ which defines it. In local coordinates this map reads

$$\Gamma : \Gamma^i \otimes \frac{\partial}{\partial \dot{q}^i} \qquad \Gamma^i = \mathrm{d}\dot{q}^i + \Gamma^i_0\,\mathrm{d}t + \Gamma^i_j\,\mathrm{d}q^j \tag{5}$$

with $\Gamma^i_0, \Gamma^i_j \in C^\infty(J^1\tau)$. Let Γ^* denote the dual endomorphism of Γ. Its image, hereafter denoted by $\mathcal{F}(J^1\tau)$, coincides with the annihilator of the horizontal distribution. Locally we have

$$\Gamma^* = \frac{\partial}{\partial \dot{q}^i} \otimes \Gamma^i$$

so that a local basis of $\mathcal{F}(J^1\tau)$ is provided by the 1-forms Γ^i.

Note in passing that $\mathcal{C}(J^1\tau) \cap \mathcal{F}(J^1\tau) = \{0\}$, $\mathcal{C}(J^1\tau) \cap \langle \mathrm{d}t\rangle = \{0\}$ and $\mathcal{F}(J^1\tau) \cap \langle \mathrm{d}t\rangle = \{0\}$. A dimensional counting thus leads to the direct sum decomposition of vector bundles

$$T^*J^1\tau = \mathcal{F}(J^1\tau) \oplus \mathcal{C}(J^1\tau) \oplus \langle \mathrm{d}t\rangle.$$

The 1-forms Γ^i, θ^i and $\mathrm{d}t$ form a local basis of $T^*J^1\tau$ adapted to this splitting.

2.2. Dynamics

Let the equations of motion of a system be defined by a second-order differential equation field ξ on $J^1\tau$. Recall that ξ is a section $\xi : J^1\tau \to J^2\tau$, viewed as a vector field on $J^1\tau$. Its coordinate expression takes the form

$$\xi = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \xi^i \frac{\partial}{\partial \dot{q}^i}$$

where $\xi^i \in C^\infty(J^1\tau)$. Locally the integral curves of ξ are the solutions of the system of differential equations

$$\ddot{q}^i = \xi^i(t, q^j, \dot{q}^j). \tag{6}$$

Alternatively, the equations of motion can be regarded as geodesic equations for an Ehresmann connection Γ on $J^1\tau \to E$ [11]. The condition to be satisfied by such a connection is that ξ takes values in the corresponding horizontal distribution, i.e. $\Gamma(\xi) = 0$. Recalling (5), we find that in local coordinates this condition reads

$$-\Gamma^i_0 - \Gamma^i_j \dot{q}^j = \xi^i. \tag{7}$$

Note that any two solutions Γ and Γ' of (7) differ for a soldering form $\sigma : J^1\tau \to T^*E \otimes V\pi$ whose coefficients σ^i_0, σ^i_j satisfy the equations $\sigma^i_0 + \sigma^i_j \dot{q}^j = 0$. Although the condition $\Gamma(\xi) = 0$ is not enough to determine the connection uniquely, there is, however, a canonical choice which has been described in many papers (see, for example, [7, 27]), so will not be repeated here. We content ourselves with giving the expressions for the coefficients Γ^i_0, Γ^i_j, which are

$$\Gamma^i_0 = -\xi^i + \frac{1}{2}\frac{\partial \xi^i}{\partial \dot{q}^j}\dot{q}^j \qquad \Gamma^i_j = -\frac{1}{2}\frac{\partial \xi^i}{\partial \dot{q}^j}. \tag{8}$$

In order to gain further insight into the meaning of a connection on $J^1\tau \to E$, let us assume that E is a four-dimensional affine bundle over $\mathbb{R}$. We denote by q^i and q'^i, $i = 1, 2, 3$, the Cartesian coordinates of a particle with respect to two frames of reference, with the corresponding transformation formula given by $q'^i = A^i_j(t)(q^j + R^j(t))$. Here A^i_j is an element of the group $SO(3)$ and R^j belongs to $\mathbb{R}^3$. Then according to (1) we have $\dot{q}'^i = A^i_j(\dot{q}^j - U^j)$ where $U^j = \omega^j_k(q^k + R^k) - \mathrm{d}R^j/\mathrm{d}t$ is the drag velocity and ω^j_k is an element of the Lie algebra of $SO(3)$. Given a connection Γ on $J^1\tau \to E$, we denote by Γ^i_0, Γ^i_j and Γ'^i_0, Γ'^i_j its connection coefficients in the two frames. From the formula for the change of the connection coefficients under a coordinate transformation (see [18], p 163), we find that

$$\Gamma'^i_j = A^i_k(\Gamma^k_h - \omega^k_h)(A^{-1})^h_j$$
$$\Gamma'^i_0 = A^i_j\left[\Gamma^j_0 + \Gamma^j_k U^k - \omega^j_k(\dot{q}^k - U^k) - V^j\right]$$

where $V^j = (\omega^2)^j_k(q^k + R^k) + \mathrm{d}\omega^j_k/\mathrm{d}t\,(q^k + R^k) - \mathrm{d}^2R^j/\mathrm{d}t^2$ is the drag acceleration. It follows from (7) that the force (per unit mass) ξ acting on the particle changes according to the well known relation

$$\xi'^i = A^i_j\left[\xi^j + 2\omega^j_k(\dot{q}^k - U^k) + \left((\omega^2)^j_k + \frac{\mathrm{d}\omega^j_k}{\mathrm{d}t}\right)(q^k + R^k) - \frac{\mathrm{d}^2R^j}{\mathrm{d}t^2}\right]. \tag{9}$$

In the subsequent discussion Γ will always be the connection defined in (8). We shall refer to it as the dynamical connection.

With the objects introduced so far, we define a 2-form on $J^1\tau$ as follows:

$$\Omega(X, Y) = g(\Gamma(X), J(Y)) - g(J(X), \Gamma(Y)) \qquad \forall\, X, Y \in \mathcal{D}(J^1\tau). \tag{10}$$

A glance at (3)–(5) leads immediately to the coordinate expression

$$\Omega = g_{ij}\Gamma^i \wedge \theta^j. \tag{11}$$

The importance of this 2-form relies in the fact that the dynamics of a mechanical system can be obtained directly from its kernel. For, it is easily seen that $X \in \operatorname{Ker}\Omega$ iff $J(X) = 0$ and $\Gamma(X) = 0$, i.e. iff X is annihilated by all contact forms and all forms in $\mathcal{F}(J^1\tau)$. In local coordinates, putting $X = X^0\partial/\partial t + X^i\partial/\partial q^i + \dot{X}^i\partial/\partial\dot{q}^i$, these conditions read

$$X^i - \dot{q}^i X^0 = 0$$

$$\dot{X}^i + \Gamma^i_0 X^0 + \Gamma^i_j X^j = 0.$$

Hence X belongs to the one-dimensional distribution on $J^1\tau$ spanned by the SODE field ξ.

Let us make some remarks on the 2-form Ω. First of all, note that it is a frame-independent attribute of a mechanical system. As is well known (see, for example, [7]), the closure of Ω ($\mathrm{d}\Omega = 0$) is locally equivalent to the existence of a Lagrangian $\mathcal{L} \in C^\infty(J^1\tau)$ such that $\Omega = \mathrm{d}\omega_{\mathcal{L}}$, where

$$\omega_{\mathcal{L}} = \mathcal{L}\,\mathrm{d}t + \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\theta^i \tag{12}$$

is the corresponding Cartan 1-form. In this case the equations of motion are equivalent to the Euler–Lagrange equations for the Lagrangian $\mathcal{L}$, that is,

$$\xi^i = g^{ij}\left(\frac{\partial\mathcal{L}}{\partial q^j} - \frac{\partial^2\mathcal{L}}{\partial\dot{q}^j\partial t} - \frac{\partial^2\mathcal{L}}{\partial\dot{q}^j\partial q^k}\dot{q}^k\right) \tag{13}$$

where g^{ij} is the inverse matrix of $g_{ij} = \partial^2\mathcal{L}/\partial\dot{q}^i\partial\dot{q}^j$.

Going back to the general case ($\mathrm{d}\Omega \neq 0$), let $\mathcal{L} \in C^\infty(J^1\tau)$ be a given regular Lagrangian and $\xi_{\mathcal{L}}$ the corresponding Lagrangian SODE field on $J^1\tau$ (see (13)). Then the SODE field ξ describing the dynamics of a system can be written as $\xi = \xi_{\mathcal{L}} + Q$, where $Q = Q^i\partial/\partial\dot{q}^i$ is a vertical vector field on $J^1\tau$ representing a dissipative force. The dynamical connection (8) takes the form $\Gamma = \Gamma_{\mathcal{L}} + \sigma$, where $\Gamma_{\mathcal{L}}$ is the dynamical connection associated with $\xi_{\mathcal{L}}$ and σ is the soldering form

$$\sigma^i_0 = -Q^i + \frac{1}{2}\frac{\partial Q^i}{\partial\dot{q}^j}\dot{q}^j \qquad \sigma^i_j = -\frac{1}{2}\frac{\partial Q^i}{\partial\dot{q}^j}.$$

A straightforward computation then leads to the following expression for Ω:

$$\Omega = \mathrm{d}\omega_{\mathcal{L}} + \Omega_Q \tag{14}$$

where Ω_Q is the semi-basic 2-form over E with the local expression

$$\Omega_Q = \frac{1}{4}\left(\frac{\partial Q_i}{\partial\dot{q}^j} - \frac{\partial Q_j}{\partial\dot{q}^i}\right)\theta^i \wedge \theta^j + Q_i\theta^i \wedge \mathrm{d}t. \tag{15}$$

Here we have put $Q_i = g_{ij}Q^j$.

One further remark is perhaps worth making here. The definition of Ω does not involve the choice of any *a priori* Lagrangian. This is an important feature, not only because it enables us to carry out the analysis in a frame-independent language, but also in view of the well known fact that the symmetries of the 2-form Ω are often greater than the symmetries of the Lagrangian.

3. Systems with constraints

3.1. Geometry of the constraint submanifold

Let $i : C \hookrightarrow J^1\tau$ be an embedded submanifold of $J^1\tau$ fibred over E with fibre dimension m. The meaning of C is that of imposing some external velocity-dependent constraints on a mechanical system. Locally the embedding i can be represented either by equations

$$\dot{q}^i = \psi^i(t, q^j, z^a) \tag{16}$$

with

$$\operatorname{rank}\left(\frac{\partial \psi^i}{\partial z^a}\right) = m \tag{17}$$

where (t, q^i, z^a) are coordinates on C, or by $n - m$ equations

$$\phi^\mu(t, q^i, \dot{q}^i) = 0 \tag{18}$$

with

$$\operatorname{rank}\left(\frac{\partial \phi^\mu}{\partial \dot{q}^i}\right) = n - m. \tag{19}$$

Following Chetaev's ideas on the extension of the concept of virtual displacement to the class of non-holonomic nonlinear constraints [24, 25], we define a virtual displacement of the system in the kinetic state $z \in C$ as a vertical vector $Y \in V_{\pi(z)}\tau$ such that its vertical lift Y^v (2) belongs to T_zC.

We denote by H the pre-image of $VC = V\pi \cap TC$ with respect to the vertical endomorphism J. It is easily seen that H is a sub-bundle of $TJ^1\tau|C$ with fibre dimension $n + m + 1$. If $z \in C$ is an admissible kinetic state of the system, then a tangent vector $X \in H_z$ can be regarded as an infinitesimal variation of z (variation of time, coordinates and velocities) whose corresponding variation $J(X)$ of the spacetime configuration $\pi(z) \in E$ is a virtual displacement in the sense of Chetaev.

Let us consider the annihilator H^o of H in $T^*J^1\tau$. We have that $H^o = J^*(TC^o)$, where TC^o is the annihilator of TC in $T^*J^1\tau$. Indeed, it can be immediately verified that $J^*(TC^o) \subset H^o$; a dimensional counting then leads to the stated equality. To obtain the coordinate description of H and H^o we use the representation (18) for the constraint submanifold C. Since the differentials $\mathrm{d}\phi^\mu$ span TC^o, it follows that H^o is generated by the linearly independent 1-forms

$$J^*(\mathrm{d}\phi^\mu) = \frac{\partial \phi^\mu}{\partial \dot{q}^i}\theta^i. \tag{20}$$

Hence a vector field $X = X^0\partial/\partial t + X^i\partial/\partial q^i + \dot{X}^i\partial/\partial \dot{q}^i$ belongs to H iff

$$\frac{\partial \phi^\mu}{\partial \dot{q}^i}(X^i - \dot{q}^i X^0) = 0. \tag{21}$$

Let K be the intersection $H \cap TC$. Due to the fact that C is fibred over E, it is easily seen that K is a $(2m + 1)$-dimensional distribution on C. Consider the annihilator K^o of K in T^*C. Following [22], we shall refer to it as the Chetaev bundle. Its sections will be called Chetaev 1-forms. Since $K = H \cap TC$, the Chetaev bundle coincides with the pull-back over C of H^o. From (20) we derive a local basis for the Chetaev forms, which is given by

$$\eta^\mu = \frac{\partial \phi^\mu}{\partial \dot{q}^i}\widehat{\theta^i} \qquad \widehat{\theta^i} = \mathrm{d}q^i - \psi^i\,\mathrm{d}t. \tag{22}$$

We shall say that the constraint C (or more precisely, the fibration $\pi : C \to E$) is integrable iff K is a (completely) integrable distribution or, equivalently, iff the ideal $\mathcal{I}$ generated by the module of Chetaev forms is a differential ideal. This, in turn, can be shown to be equivalent to the property that C admits at least one local representation (18) of the special form [22]

$$\phi^\mu = \frac{\partial\varphi^\mu}{\partial t} + \frac{\partial\varphi^\mu}{\partial q^i}\dot{q}^i = 0$$

with $\varphi^\mu \in C^\infty(E)$. Consequently, every leaf of K can be locally represented as the jet manifold of an $(m+1)$-dimensional submanifold of E fibred over $\mathbb{R}$.

Let $\widehat{g}$ be the symmetric bilinear form on VC obtained by evaluating the fibre metric g on vertical vectors which are tangent to C. By means of $\widehat{g}$ we can define an orthogonal projection $P : V\pi|C \to VC$ according to the relation

$$\widehat{g}(P(X), Y) = g(X, Y) \qquad \forall X \in V_z\pi \quad Y \in V_zC$$

and also the direct sum decomposition

$$V\pi|C = VC \oplus VC^\perp \tag{23}$$

where $VC^\perp$ is the kernel of P. Accordingly, each vertical vector $X \in V_z\pi$ can be written in the form $X = X' + X^\perp$ with $X' \in V_zC$ and $X^\perp \in V_zC^\perp$. Using (16) we find the coordinate expressions

$$g_{ab} = \widehat{g}\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}\right) = \frac{\partial\psi^i}{\partial z^a}\frac{\partial\psi^j}{\partial z^b}g_{ij} \tag{24}$$

$$P\left(\frac{\partial}{\partial\dot{q}^i}\right) = \Lambda^a_i\frac{\partial}{\partial z^a} \qquad \Lambda^a_i = g^{ab}\frac{\partial\psi^j}{\partial z^b}g_{ij} \tag{25}$$

with $g^{ac}g_{cb} = \delta^a_b$.

Owing to the projection P, the vertical endomorphism J induces a map $\widehat{J} : TC \to TC$ according to the relation

$$\widehat{J}(X) = P \circ J(X) \qquad \forall\, X \in \mathcal{D}(C).$$

From (4) and (25) we find the local expression

$$\widehat{J} = \theta^a \otimes \frac{\partial}{\partial z^a} \qquad \theta^a = \Lambda^a_i\widehat{\theta}^i. \tag{26}$$

Let $\widehat{J}^* : T^*C \to T^*C$ be the dual endomorphism. Its image is a vector sub-bundle of T^*C, hereafter denoted by V^*C. From (26) we see that θ^a is a local basis of V^*C.

Lemma 1 (see [22]). *The bundle $\mathcal{C}(C)$ of contact 1-forms on C decomposes into a direct sum of vector bundles, namely,*

$$\mathcal{C}(C) = V^*C \oplus K^o. \tag{27}$$

Proof. We show that $V^*C \cap K^o = \{0\}$. A dimensionality argument will then complete the proof. Since V^*C is the image of $\widehat{J}^*$ it coincides with the annihilator in T^*C of Ker $\widehat{J}$. This latter, in turn, is the set of vectors $X \in TC$ such that $J(X) \in VC^\perp$. On the other hand, by definition K^o is the annihilator of K, which is the set of vectors $X \in TC$ such that $J(X) \in VC$. It follows that a 1-form $\omega \in V^*C \cap K^o$ annihilates all vectors in TC, hence $\omega = 0$. □

Note that the above proof (unlike that given in [22]) does not make use of the non-degeneracy of the fibre metric. In fact, the decomposition (27) takes place under milder regularity assumptions on g, namely, the non-degeneracy of $\widehat{g}$.

For later purposes, we give here the decomposition of the contact forms $\widehat{\theta^i}$ according to lemma 1. Certainly, we can write $\widehat{\theta^i} = R^i_a\theta^a + S^i_\mu\eta^\mu$ for some coefficients $R^i_a, S^i_\mu \in C^\infty(C)$. Recalling that $\theta^a = \Lambda^a_i\widehat{\theta^i}$ and $\eta^\mu = \partial\phi^\mu/\partial\dot{q}^i\widehat{\theta^i}$, these coefficients are determined by the relation

$$R^i_a\Lambda^a_j + S^i_\mu\frac{\partial\phi^\mu}{\partial\dot{q}^j} = \delta^i_j. \tag{28}$$

Multipling both sides of this relation by $\partial\psi^j/\partial z^b$ and summing over j we find $R^i_a = \partial\psi^i/\partial z^a$, so that

$$\widehat{\theta^i} = \frac{\partial\psi^i}{\partial z^a}\theta^a + S^i_\mu\eta^\mu. \tag{29}$$

Paralleling the procedure followed for the vertical endomorphism, a connection Γ on $J^1\tau \to E$ restricts to a connection on the bundle $C \to E$ according to the relation

$$\widehat{\Gamma}(X) = P \circ \Gamma(X) \qquad \forall\, X \in \mathcal{D}(C).$$

From (5) and (25) we find the coordinate expression

$$\widehat{\Gamma} = \Gamma^a \otimes \frac{\partial}{\partial z^a} \qquad \Gamma^a = \Lambda^a_i\widehat{\Gamma}^i = \mathrm{d}z^a + \Gamma^a_0\,\mathrm{d}t + \Gamma^a_i\,\mathrm{d}q^i \tag{30}$$

with

$$\Gamma^a_0 = \Lambda^a_i\left(\frac{\partial\psi^i}{\partial t} + \Gamma^i_0\right) \qquad \Gamma^a_j = \Lambda^a_i\left(\frac{\partial\psi^i}{\partial q^j} + \Gamma^i_j\right). \tag{31}$$

Let $\widehat{\Gamma}^* : T^*C \to T^*C$ be the dual of $\widehat{\Gamma}$. Its image is a sub-bundle of T^*C denoted by $\mathcal{F}(C)$, with local basis provided by the 1-forms Γ^a.

In summary, we have that T^*C decomposes into a direct sum of vector bundles, namely,

$$T^*C = \mathcal{F}(C) \oplus V^*C \oplus K^o \oplus \langle\mathrm{d}t\rangle. \tag{32}$$

For, the connection $\widehat{\Gamma}$ splits T^*C into the sum of $\mathcal{F}(C)$ and the bundle of semi-basic forms over E. This latter, in turn, decomposes into the direct sum of the contact bundle over C and $\langle\mathrm{d}t\rangle$. Finally, lemma 1 leads to the four-way split (32). From previous considerations it is also clear that the 1-forms

$$\Gamma^a, \theta^a, \eta^\mu, \mathrm{d}t \tag{33}$$

provide a local basis of T^*C adapted to this decomposition. A straightforward computation shows that a 1-form $\omega = \omega_0\,\mathrm{d}t + \omega_i\,\mathrm{d}q^i + \omega_a\,\mathrm{d}z^a$ on C can be written in terms of this basis as follows:

$$\omega = \omega_a\Gamma^a + (\omega_i - \Gamma^b_i\omega_b)\frac{\partial\psi^i}{\partial z^a}\theta^a + (\omega_i - \Gamma^b_i\omega_b)S^i_\mu\eta^\mu + [\omega_0 + \omega_i\psi^i - \omega_a(\Gamma^a_0 + \Gamma^a_i\psi^i)]\,\mathrm{d}t. \tag{34}$$

3.2. *Constrained dynamics*

Let $\mathrm{orth}(H)$ be the sub-bundle of $TJ^1\tau|C$ which is orthogonal to H with respect to Ω, that is,

$$X \in \mathrm{orth}(H) \qquad \text{iff} \quad \Omega(X, Y) = 0 \quad \forall\, Y \in H.$$

The results of the previous section are completed by the following theorem on the dynamics of a constrained system.

Theorem 1. *The intersection $TC \cap \mathrm{orth}(H)$ is a one-dimensional distribution on C spanned by a (unique) SODE field $\widehat{\xi}$ on C (i.e. a vector field $\widehat{\xi}$ on C such that $\widehat{\xi}\rfloor \mathrm{d}t = 1$ and $J(\widehat{\xi}) = 0$).*

Proof. From the definition of Ω, taking into account that $V\pi|C \subset H$, we have that $X \in \mathrm{orth}(H)$ implies $J(X) = 0$. Hence $\Omega(X, Y) = g(\Gamma(X), J(Y)) = 0$ for all $Y \in H$, so that $\Gamma(X)' = 0$. It follows that $X \in TC \cap \mathrm{orth}(H)$ iff

$$J(X) = 0 \qquad \widehat{\Gamma}(X) = 0 \tag{35}$$

i.e. iff X is annihilated by all contact forms on C and all forms in $\mathcal{F}(C)$. Recalling (32) we conclude that $TC \cap \mathrm{orth}(H)$ is a one-dimensional distribution. Consequently, a SODE field $\widehat{\xi}$ on C which spans $TC \cap \mathrm{orth}(H)$, if it exists, is necessarily unique.

To prove the existence of $\widehat{\xi}$, we consider a coordinate neighbourhood in which C is represented by equations (17). On this neighbourhood we can certainly find such a SODE field $\widehat{\xi}$. This is the element of the basis dual to (33) with $\widehat{\xi}\rfloor \mathrm{d}t = 1$. Actually, if $X = X^0\partial/\partial t + X^i\partial/\partial q^i + X^a\partial/\partial z^a$ is the local expression of a vector field $X \in \mathcal{D}(C)$, then conditions (35) read

$$X^i - \psi^i X^0 = 0$$
$$X^a + \Gamma^a_0 X^0 + \Gamma^a_j X^j = 0.$$

Setting $X^0 = 1$ we find

$$\widehat{\xi} = \frac{\partial}{\partial t} + \psi^i\frac{\partial}{\partial q^i} - (\Gamma^a_0 + \Gamma^a_j\psi^j)\frac{\partial}{\partial z^a}. \tag{36}$$

Now, a partition of unity argument shows that we may find a global SODE field over C by gluing together these local solutions. □

It is perhaps worth emphasizing that the condition $X \in \mathrm{orth}(H)$ is equivalent to $J(X) = 0$ and $\Gamma(X)' = 0$. The first condition implies that, as in the non-constrained case, X belongs to the $(n + 1)$-dimensional distribution spanned by $J^2\tau$. The condition $\Gamma(X)' = 0$ expresses the principle of virtual work. More precisely, if $X \in T_zC$ is an infinitesimal possible evolution of the system from the kinetic state $z \in C$, then $\Gamma(X)$ is the force that is needed in order for the non-constrained system to undergo that infinitesimal evolution. The quantity $g(\Gamma(X), J(Y))$ is the elementary work performed by the force of constraint on the Chetaev virtual displacement $J(Y)$. Thus the condition $\Gamma(X)' = 0$ means that the constraints do not work on the class of chosen virtual displacements. Theorem 1 therefore shows in what way the principle of virtual work makes the constrained dynamics determined; the motions of a system with ideal constraints (in the Chetaev sense) are the integral curves of the unique SODE field on C which spans the one-dimensional distribution $TC\cap\mathrm{orth}(H)$. Note also that, as in the non-constrained case, these curves can be regarded as geodesics of the connection $\widehat{\Gamma}$ on the bundle $C \to E$ induced by the dynamical connection (8).

Now let $\widehat{\Omega}$ be the pull-back of the 2-form Ω by i and let $\mathrm{orth}(K)$ be the orthogonal sub-bundle of the distribution K with respect to $\widehat{\Omega}$.

Theorem 2. $K \cap \mathrm{orth}(K) = TC \cap \mathrm{orth}(H)$.

Proof. Taking into account (23), the evaluation of Ω on a pair of vector fields $X, Y \in \mathcal{D}(J^1\tau)$ reads

$$\Omega(X, Y) = g(\Gamma(X)', J(Y)') - g(J(X)', \Gamma(Y)') + g(\Gamma(X)^{\perp}, J(Y)^{\perp}) - g(J(X)^{\perp}, \Gamma(Y)^{\perp}). \tag{37}$$

Recalling that a vector field $X \in \mathcal{D}(C)$ is a section of K iff $J(X)^{\perp} = 0$, we have $\widehat{\Omega}(X, Y) = \widehat{g}(\widehat{\Gamma}(X), \widehat{J}(Y)) - \widehat{g}(\widehat{J}(X), \widehat{\Gamma}(Y))$ for every pair of vector fields $X, Y \in \mathcal{D}(C)$ taking values into K. Using the arbitrariness of $Y \in K$ and proceeding as in the proof of theorem 1, we find that $X \in K \cap \mathrm{orth}(K)$ iff $J(X)^{\perp} = 0$, $\widehat{J}(X) = 0$ and $\widehat{\Gamma}(X) = 0$, that is, iff conditions (35) hold. □

Some remarks derived from theorem 2 are worth making here. Recall that if S is a submanifold of C and Ω_S is the pull-back of $\widehat{\Omega}$ by the natural inclusion, then $\mathrm{Ker}(\Omega_S) = TS \cap \mathrm{orth}(TS)$. Therefore, if S is an integral manifold of the distribution K we have the equality $K \cap \mathrm{orth}(K) = \mathrm{Ker}(\Omega_S)$. Hence the portion of the constrained dynamics lying on S is described by the kernel of the 2-form Ω_S. This means that if C, in particular, is integrable, all properties of the non-constrained motion also apply to the constrained one. On the other hand, if C is not integrable the motion of the constrained system is still defined by a one-dimensional distribution, namely, $K \cap \mathrm{orth}(K)$. However, this is no longer in general the kernel of a 2-form induced only by Ω.

The following result, which also extends theorem 1 of [30] to the class of non-Lagrangian systems, is an immediate consequence of the previous theorems.

Corollary 1. *There exists a unique vector field $\widehat{\xi}$ on C such that $\widehat{\xi} \rfloor \mathrm{d}t = 1$, $\widehat{\xi} \in K$ and $\widehat{\xi} \rfloor \widehat{\Omega} \in K^o$. In addition, $\widehat{\xi}$ is then necessarily a SODE field on C.*

To close this section we now study further properties of $\widehat{\Omega}$. First of all, we give its expression in terms of the basis (33). In accordance with (34), we write the 1-forms $\widehat{\Gamma}^i = i^*\Gamma^i$ as follows:

$$\widehat{\Gamma}^i = \frac{\partial \psi^i}{\partial z^a}\Gamma^a + \widetilde{\Gamma}^i$$

where $\widetilde{\Gamma}^i$ are semi-basic forms over E whose explicit form is of no importance here. Recalling (11) and (29) we obtain

$$\widehat{\Omega} = g_{ij}\left(\frac{\partial \psi^i}{\partial z^a}\Gamma^a + \widetilde{\Gamma}^i\right) \wedge \left(\frac{\partial \psi^j}{\partial z^b}\theta^b + S^j_{\mu}\eta^{\mu}\right).$$

By means of (28) and the relation $\Lambda^a_i\widetilde{\Gamma}^i = \Lambda^a_i(\widehat{\Gamma}^i - \Gamma^a) = 0$, two terms on the right-hand side vanish so that we find

$$\widehat{\Omega} = g_{ab}\Gamma^a \wedge \theta^b + g_{ij}S^j_{\mu}\widetilde{\Gamma}^i \wedge \eta^{\mu}. \tag{38}$$

Next, we study the condition

$$\mathrm{d}\widehat{\Omega} \in \mathrm{d}\mathcal{I} = \{\mathrm{d}\omega : \omega \in \mathcal{I}\}. \tag{39}$$

This condition will play a role in the subsequent discussion on symmetries and invariants of a non-holonomic system. Also it characterizes the class of SODE fields $\widehat{\xi}$ on C for which there exists a non-holonomic Lagrangian [23] in the following sense.

Theorem 3. *Let $\widehat{\Omega}$ be the 2-form (38) and $\widehat{\xi}$ the corresponding SODE field on C according to corollary 1. If condition (39) holds then there exists, at least locally, a function $l \in C^{\infty}(C)$ and a Chetaev form $\eta \in K^o$ such that $\widehat{\Omega} - \mathrm{d}\beta \in \mathcal{I}$ where β is given by*

$$\beta = l\,\mathrm{d}t + \frac{\partial l}{\partial z^a}\theta^a + \eta.$$

Proof. Condition (39) ensures that there exists locally a 1-form α such that $\widehat{\Omega} = \mathrm{d}\alpha + \rho$ for some form $\rho \in \mathcal{I}$. Moreover, since the evaluation of $\widehat{\Omega}$ on every pair of vertical fields vanishes, we can find locally a function $G \in C^\infty(C)$ such that $X\rfloor\alpha = X(G)$ for all $X \in VC$. Putting $\beta = \alpha - \mathrm{d}G$ we have $X\rfloor\beta = 0$ for all $X \in VC$, so that in terms of the basis (33) β takes the form

$$\beta = \beta_a\theta^a + \beta_\mu\eta^\mu + (\widehat{\xi}\rfloor\beta)\,\mathrm{d}t.$$

By defining $l = \widehat{\xi}\rfloor\beta$ we see that

$$\widehat{\xi}\rfloor\widehat{\Omega} = L_{\widehat{\xi}}\beta - \mathrm{d}l + \widehat{\xi}\rfloor\rho.$$

Writing out this relation in terms of the basis (33) and equating the coefficients of Γ^a we obtain $\beta^a = \partial l/\partial z^a$. This completes the proof of the theorem. □

A glance at (38) shows that (39) is satisfied if $g_{ab}\Gamma^a \wedge \theta^b$ is closed. Note also that (39) is trivially satisfied if $\mathrm{d}\Omega = 0$. As we noted in the previous section, this is (locally) equivalent for the non-constrained system to admit a Lagrangian formulation. If $\mathcal{L} \in C^\infty(J^1\tau)$ is a Lagrangian, then the 1-form β is given by

$$\widehat{\omega}_{\mathcal{L}} = \widehat{\mathcal{L}}\,\mathrm{d}t + \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\widehat{\theta}^i = \widehat{\mathcal{L}}\,\mathrm{d}t + \frac{\partial\widehat{\mathcal{L}}}{\partial z^a}\theta^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^i}S^i_\mu\eta^\mu.$$

4. Symmetries and first integrals

Motivated by [5], in this section we consider a set of vector fields on C which directly generate first integrals of the SODE field $\widehat{\xi}$. These vector fields are related to a generalized version of Noether's theorem and its converse for non-conservative systems with non-holonomic constraints (see, e.g., [16]).

4.1. Generating first integrals

To begin with, let us introduce some preliminary definitions.

Definition 1. *$Z \in \mathcal{D}(C)$ is a dynamical symmetry iff $[Z, \widehat{\xi}] = h\widehat{\xi}$ with $h \in C^\infty(C)$.*

$Z \in \mathcal{D}(C)$ is a trivial symmetry iff $Z = h\widehat{\xi}$ with $h \in C^\infty(C)$ or, equivalently, iff Z is a section of $K \cap \mathrm{orth}(K)$.

$Z \in \mathcal{D}(C)$ is a conformal symmetry iff $L_Z K^o \subset K^o$ and $L_Z\widehat{\Omega} - k\widehat{\Omega} \in \mathcal{I}$ for some function $k \in C^\infty(C)$, where $\mathcal{I}$ is the ideal generated by the module of Chetaev forms. In particular, we call Z a symmetry if the latter condition is replaced by $L_Z\widehat{\Omega} \in \mathcal{I}$.

Corollary 2.

(a) $Z \in \mathcal{D}(C)$ is a dynamical symmetry iff $[Z, \widehat{\xi}] \in K$ and $[Z, \widehat{\xi}]\rfloor\widehat{\Omega} \in K^o$.
(b) Each conformal symmetry is a dynamical symmetry.

The proof of (a) is an immediate consequence of corollary 1. Statement (b) follows from (a) by noticing that for each $Z \in \mathcal{D}(C)$, $L_Z K^o \subset K^o$ iff $L_z K \subset K$, and by using the formula [17]

$$[Z, \widehat{\xi}]\rfloor\widehat{\Omega} = L_Z(\widehat{\xi}\rfloor\widehat{\Omega}) - \widehat{\xi}\rfloor L_Z\widehat{\Omega}. \qquad (40)$$

We now consider the set of vector fields $Z \in \mathcal{D}(C)$ such that

$$Z \in K \qquad (41)$$

$$\mathrm{d}(Z\rfloor\widehat{\Omega}) \in \mathrm{d}K^o = \{\mathrm{d}\eta : \eta \in K^o\}. \qquad (42)$$

We shall denote this set by $\mathcal{V}$. As is easily seen, $\mathcal{V}$ is a linear space over $\mathbb{R}$ which contains all trivial symmetries of $\widehat{\xi}$.

By Poincaré's lemma, for each $Z \in \mathcal{V}$ there exists (at least locally) a function $F \in C^\infty(C)$ which is related to Z according to the relation

$$Z \rfloor \widehat{\Omega} - \mathrm{d}F \in K^o. \tag{43}$$

Then corollary 1 implies that $\widehat{\xi}(F) = 0$, i.e. F is a first integral of $\widehat{\xi}$. Conversely, we have the following theorem.

Theorem 4. *Let ω be a 1-form on C. Then $\widehat{\xi} \rfloor \omega = 0$ iff $Z \rfloor \widehat{\Omega} - \omega \in K^o$ for some vector field $Z \in \mathcal{D}(C)$ taking values into K. In particular, for every first integral $F \in C^\infty(C)$ of $\widehat{\xi}$ there exists a vector field $Z \in K$ such that condition (43) holds, hence $Z \in \mathcal{V}$.*

Proof. From corollary 1, it is clear that if $Z \rfloor \widehat{\Omega} - \omega \in K^o$ for some vector field $Z \in K$ then $\widehat{\xi} \rfloor \omega = 0$.

Conversely, if $\widehat{\xi} \rfloor \omega = 0$ then ω takes the form $\omega = B_a \Gamma^a + C_a \theta^a + D_\mu \eta^\mu$ for some coefficients B_a, C_a, D_μ. From (38), the equation $Z \rfloor \widehat{\Omega} - \omega \in K^o$ splits up into the following system:

$$Z \rfloor \theta^a = -g^{ab} B_b \tag{44}$$

$$Z \rfloor \Gamma^a = g^{ab} C_b \tag{45}$$

$$Z \rfloor \eta^\mu = 0. \tag{46}$$

It follows that we can find a solution $Z \in K$ in a neighbourhood of each point of C. Now a partition of unity argument enables us to construct a global solution. □

Note that for a given $Z \in \mathcal{V}$ the corresponding invariant is not uniquely determined. If F and F' are two integrals associated with the same vector field Z then $\mathrm{d}(F - F') \in K^o$. Conversely, it follows from (43) and corollary 1 that for a given invariant F of $\widehat{\xi}$ the vector field $Z \in \mathcal{V}$ will be determined up to a trivial symmetry of $\widehat{\xi}$. This can be seen also in local coordinates using (44)–(46). By replacing the 1-form ω in these expressions with the differential $\mathrm{d}F$ of an invariant and writing out $\mathrm{d}F$ in terms of the basis (33), straightforward algebra leads to

$$Z^i = \psi^i Z^0 - \frac{\partial \psi^i}{\partial z^a} g^{ab} \frac{\partial F}{\partial z^b} \tag{47}$$

$$Z^a = -(\Gamma^a_0 + \Gamma^a_j \psi^j) Z^0 + \Gamma^a_i \frac{\partial \psi^i}{\partial z^c} g^{cb} \frac{\partial F}{\partial z^b} + g^{ab} \left(\frac{\partial F}{\partial q^i} - \Gamma^c_i \frac{\partial F}{\partial z^c} \right) \frac{\partial \psi^i}{\partial z^b}. \tag{48}$$

The above considerations show that there is (at least locally) one-to-one correspondence between the set of equivalence classes of vector fields in $\mathcal{V}$ and the set of equivalence classes of invariants, where two vector fields in $\mathcal{V}$ are equivalent if they differ by a section of $K \cap \mathrm{orth}(K)$, whereas two invariants are identified if their differentials differ for a Chetaev form.

4.2. Choosing a Lagrangian

In practical applications we are given a regular Lagrangian $\mathcal{L} \in C^\infty(J^1\tau)$ and a dissipative force Q. Then the 2-form Ω takes the form (14), so that $\widehat{\Omega} = \mathrm{d}\widehat{\omega}_{\mathcal{L}} + \widehat{\Omega}_Q$. Condition (42) becomes

$$\mathrm{d}(L_Z \widehat{\omega}_{\mathcal{L}} + Z \rfloor \widehat{\Omega}_Q) \in \mathrm{d}K^o$$

and applying Poincaré's lemma we see that (at least locally)

$$L_Z\widehat{\omega}_{\mathcal{L}} + Z\rfloor\widehat{\Omega}_Q - \mathrm{d}f \in K^o \tag{49}$$

for some function $f \in C^\infty(C)$.

Let us write equation (49) in local coordinates. Putting $Z = Z^0\partial/\partial t + Z^i\partial/\partial q^i + Z^a\partial/\partial z^a$, using (12) and (15) and writing out (49) in terms of the basis (33), we find the following set of partial differential equations for Z^0, Z^j, Z^a and f:

$$(\widehat{\mathcal{L}} - \pi_j\psi^j)\frac{\partial Z^0}{\partial z^a} + \pi_j\frac{\partial Z^j}{\partial z^a} = \frac{\partial f}{\partial z^a} \tag{50}$$

$$\begin{aligned}\Bigg[Z(\pi_j) + (\widehat{\mathcal{L}} - \pi_i\psi^i)\left(\frac{\partial Z^0}{\partial q^j} - \Gamma^b_j\frac{\partial Z^0}{\partial z^b}\right) + \pi_i\left(\frac{\partial Z^i}{\partial q^j} - \Gamma^b_j\frac{\partial Z^i}{\partial z^b}\right)\\ + \frac{1}{2}\left(\frac{\partial Q_i}{\partial \dot q^j} - \frac{\partial Q_j}{\partial \dot q^i}\right)(Z^i - \psi^i Z^0) - Q_j Z^0\Bigg]\frac{\partial \psi^j}{\partial z^a} = \left(\frac{\partial f}{\partial q^j} - \Gamma^b_j\frac{\partial f}{\partial z^b}\right)\frac{\partial \psi^j}{\partial z^a}\end{aligned} \tag{51}$$

$$Z(\widehat{\mathcal{L}}) - \pi_j Z(\psi^j) + (\widehat{\mathcal{L}} - \pi_j\psi^j)\widehat{\xi}(Z^0) + \pi_j\widehat{\xi}(Z^j) + Q_i(Z^i - \psi^i Z^0) = \widehat{\xi}(f). \tag{52}$$

Here we have the set $\pi_j = i^*(\partial\mathcal{L}/\partial\dot q^j)$. Note that (50) and (52) do not involve the coefficients Z^a. Moreover, whenever (Z^0, Z^i, f) is a solution of these equations, (51) immediately furnishes the remaining components Z^a. Equations (50) and (52) are the so-called generalized Killing equations for Noether symmetries of non-conservative non-holonomic systems.

Further insight in the structure of solutions of (50)–(52) can be obtained by considering the formula (40), which in the present case becomes after straightforward algebra

$$[Z, \widehat{\xi}]\rfloor\widehat{\Omega} = L_Z(\widehat{\xi}\rfloor\widehat{\Omega}) - L_{\widehat{\xi}}(Z\rfloor\widehat{\Omega}) - \widehat{\xi}\rfloor Z\rfloor\,\mathrm{d}\widehat{\Omega}_Q.$$

Writing out this equality in terms of the basis (33) and using (38) and (15), we obtain by equating the coefficients of Γ^a,

$$\begin{aligned}g_{ab}\Lambda^b_i\left[Z(\psi^i) - \widehat{\xi}(Z^i) + \psi^i\widehat{\xi}(Z^0)\right] = -(\widehat{\xi}\rfloor\widehat{\Omega})_\mu\frac{\partial\phi^\mu}{\partial\dot q^i}\left(\frac{\partial Z^i}{\partial z^a} - \psi^i\frac{\partial Z^0}{\partial z^a}\right)\\ + \frac{1}{2}\left(\frac{\partial Q_j}{\partial\dot q^i} + \frac{\partial Q_i}{\partial\dot q^j}\right)\frac{\partial\psi^i}{\partial z^a}(Z^j - \psi^j Z^0).\end{aligned} \tag{53}$$

As is easily seen, this relation enables us to determine the components Z^a of Z in terms of Z^0 and Z^i only. Therefore, it can be used in substitution of equation (51). Note that if Z is a projectable vector field onto E (i.e. Z^0 and Z^i do not depend on velocities), then the term $\Lambda^a_i\left[Z(\psi^i) - \widehat{\xi}(Z^i) + \psi^i\widehat{\xi}(Z^0)\right]$ on the left-hand side of (53) can be written more concisely as $P(Z - \dot Z)$, where $\dot Z$ denotes the natural lifting on $J^1\tau$ of the projection of Z.

Rewriting (49) we obtain

$$Z\rfloor\,\mathrm{d}\widehat{\omega}_{\mathcal{L}} + Z\rfloor\widehat{\Omega}_Q - \mathrm{d}(f - Z\rfloor\widehat{\omega}_{\mathcal{L}}) \in K^o.$$

With (14) this becomes

$$Z\rfloor\widehat{\Omega} - \mathrm{d}(f - Z\rfloor\widehat{\omega}_{\mathcal{L}}) \in K^o$$

and a comparison with (43) shows that, up to a function whose differential belongs to the Chetaev bundle, the invariant generated by Z is given by

$$F = f - Z\rfloor\widehat{\omega}_{\mathcal{L}} = f - \widehat{\mathcal{L}}Z^0 - \pi_i(Z^i - \psi^i Z^0). \tag{54}$$

Hence, whenever Z and f is a solution of (50), (52) and (46), equation (54) yields a first integral of $\widehat{\xi}$.

4.3. Noether's theorem

In the previous discussion we saw that the components Z^0 and Z^i of a vector field $Z \in \mathcal{V}$ satisfy equations (50) and (52), for some function f. Here we prove the converse of this statement.

Theorem 5. *If Z^0, Z^i and f satisfy equations (50), (52) and (46), then with Z^a given by (53) we obtain a vector field $Z \in \mathcal{V}$.*

Proof. A simple computation shows that (50) and (52) can be rewritten in a concise form as follows:

$$\frac{\partial}{\partial z^a} \rfloor (L_Z \widehat{\omega}_{\mathcal{L}} - \mathrm{d}f) = 0 \tag{55}$$

$$\widehat{\xi}(f - Z \rfloor \widehat{\omega}_{\mathcal{L}}) = 0. \tag{56}$$

From (55) it immediately follows that

$$\frac{\partial}{\partial z^a} \rfloor (Z \rfloor \widehat{\Omega} - \mathrm{d}F) = 0$$

where $F = f - Z \rfloor \widehat{\omega}_{\mathcal{L}}$ is a first integral of $\widehat{\xi}$ in view of (56). This equation in turn, along with (46), leads to (47). On the other hand, we know from theorem 4 that a vector field $\widetilde{Z} \in \mathcal{V}$ exists which satisfies

$$\widetilde{Z} \rfloor (\mathrm{d}\widehat{\omega}_{\mathcal{L}} + \widehat{\Omega}_Q) - \mathrm{d}F \in K^o.$$

We now prove that $Z = \widetilde{Z} - h\widehat{\xi}$ for some function h, so that Z belongs to $\mathcal{V}$.

Since $\widetilde{Z}$ also satisfies (47), we have that $\widetilde{Z}^i - \psi^i \widetilde{Z}^0 = Z^i - \psi^i Z^0$. Thus defining a function $h = \widetilde{Z}^0 - Z^0$, we can write $Z^i = \widetilde{Z}^i - \psi^i h$. Taking into account that both Z^a and $\widetilde{Z}^a$ satisfy (53) in terms of Z^0, Z^i and $\widetilde{Z}^0$, $\widetilde{Z}^i$, respectively, an immediate computation gives $Z^a = \widetilde{Z}^a + (\Gamma^a_0 + \Gamma^a_j \psi^j)h$. This completes the proof that $Z = \widetilde{Z} - h\widehat{\xi}$. □

By virtue of the equivalence established in this section, in practical applications the problem of finding a vector field $Z \in \mathcal{V}$ may therefore be approached by solving equations (50), (52) and (46).

4.4. Further properties of vector fields in $\mathcal{V}$

The vector fields in $\mathcal{V}$ are not, in general, dynamical symmetries. However, assume that C is integrable and that condition (39) holds. Then we have

$$L_Z \widehat{\Omega} = \mathrm{d}(Z \rfloor \widehat{\Omega}) + Z \rfloor \mathrm{d}\widehat{\Omega} \in \mathcal{I} \tag{57}$$

so that each $Z \in \mathcal{V}$ is a symmetry, and hence a dynamical symmetry by virtue of corollary 2, (b). Actually, let $i_S : S \hookrightarrow C$ be an integral manifold of K and $\Omega_S = i_S^* \widehat{\Omega}$. From (57) and theorem 3 we find $L_Z \Omega_S = i_S^* L_Z \widehat{\Omega} = 0$ with

$$\Omega_S = i_S^*(\mathrm{d}\beta + \rho) = \mathrm{d}\beta_S = \mathrm{d}\left(l_S \, \mathrm{d}t + \frac{\partial l_S}{\partial \dot{q}^a} \theta^a \right)$$

and $l_S = i_S^* l$. Here the velocities $\dot{q}^a$ play the role of the coordinates z^a on S. Hence, in the case where C is integrable and $\mathrm{d}\widehat{\Omega} \in \mathrm{d}\mathcal{I}$ we recover the characterization of Noether symmetry for holonomic systems.

Another consequence of these assumptions concerns the Lie algebra structure of $\mathcal{V}$. In general, $\mathcal{V}$ is not closed under the Lie bracket. In fact, for each $Z, Z' \in \mathcal{V}$ we have

$$[Z, Z']\rfloor\widehat{\Omega} = L_Z(Z'\rfloor\widehat{\Omega}) - Z'\rfloor L_Z\widehat{\Omega}$$
$$= L_Z(Z'\rfloor\widehat{\Omega}) - Z'\rfloor Z\rfloor \mathrm{d}\widehat{\Omega} - Z'\rfloor \mathrm{d}(Z\rfloor\widehat{\Omega}).$$

By taking the exterior derivative of both sides we obtain

$$\mathrm{d}([Z, Z']\rfloor\widehat{\Omega}) = L_Z\,\mathrm{d}(Z'\rfloor\widehat{\Omega}) - L_{Z'}\,\mathrm{d}(Z\rfloor\widehat{\Omega}) - \mathrm{d}(Z'\rfloor Z\rfloor\,\mathrm{d}\widehat{\Omega})$$

and we see that the right-hand side belongs to $\mathrm{d}K^o$ if C is integrable and $\mathrm{d}\widehat{\Omega} \in \mathrm{d}\mathcal{I}$. Thus $\mathcal{V}$ is closed under the Lie bracket.

Going back to the general case, suppose now that the SODE field $\widehat{\xi}$ (36) satisfies the conditions

$$L_{\widehat{\xi}}K^o \subset K^o \tag{58}$$

$$L_{\widehat{\xi}}\widehat{\Omega} - k\widehat{\Omega} \in \mathcal{I} \tag{59}$$

with $k \in C^\infty(C)$, i.e. $\widehat{\xi}$ is a conformal symmetry of $\widehat{\Omega}$. Putting $l = \exp(h)$, with $h \in C^\infty(C)$ a (local) solution of $\widehat{\xi}(h) = k$, we obtain

$$\widehat{\xi}(l) = kl. \tag{60}$$

Then we have the following theorem.

Theorem 6. *For each $Z \in \mathcal{V}$ the vector field lZ is a dynamical symmetry of $\widehat{\xi}$. If, in particular, $\widehat{\xi}$ is a symmetry (i.e. $k = 0$), then every $Z \in \mathcal{V}$ is a dynamical symmetry.*

Proof. Condition (58) immediately implies $\left[lZ, \widehat{\xi}\right] \in K$. Moreover, we have

$$\left[lZ, \widehat{\xi}\right]\rfloor\widehat{\Omega} = L_{lZ}(\widehat{\xi}\rfloor\widehat{\Omega}) - \widehat{\xi}\rfloor L_{lZ}\widehat{\Omega}$$
$$= lZ\rfloor L_{\widehat{\xi}}\widehat{\Omega} - L_{\widehat{\xi}}(lZ\rfloor\widehat{\Omega})$$
$$= (-\widehat{\xi}(l) + kl)Z\rfloor\widehat{\Omega} + \text{a Chetaev form}$$

and a glance at (60) shows that $\left[lZ, \widehat{\xi}\right]\rfloor\widehat{\Omega} \in K^o$. Hence lZ is a dynamical symmetry by virtue of corollary 2, (a).

If $\widehat{\xi}$ is a symmetry then the function l is a first integral of $\widehat{\xi}$. It follows that $\left[lZ, \widehat{\xi}\right] = l\left[Z, \widehat{\xi}\right]$, and hence Z is a dynamical symmetry since lZ is. □

We conclude this section by giving a property of $\mathcal{V}$ concerning its relation with symmetry vector fields.

Theorem 7. *If $Y \in \mathcal{D}(C)$ is a symmetry, then it leaves $\mathcal{V}$ invariant, i.e. $[Y, Z] \in \mathcal{V}$ for each $Z \in \mathcal{V}$. Moreover, if F is a first integral of $\widehat{\xi}$ generated by Z then $Y(F)$ is the corresponding first integral generated by $[Y, Z]$.*

Proof. Since Y is a symmetry it leaves K invariant, hence $[Y, Z] \in K$ for each $Z \in \mathcal{V}$. Moreover, from the relation

$$[Y, Z]\rfloor\widehat{\Omega} = L_Y(Z\rfloor\widehat{\Omega}) - Z\rfloor L_Y\widehat{\Omega}$$

we see that if F is the invariant corresponding to Z, then $[Y, Z]$ satisfies condition (43) with $Y(F)$ as the corresponding invariant. □

5. Examples

Example 1. Let Σ be a vector field on E with coordinate expression $\Sigma = \partial/\partial t + \Sigma^i \partial/\partial q^i$ and let Z^0 be a function of time t. As is well known (see, for example, [10], p 208), with a complete (vector field) Σ we have a trivialization of E, and hence a reference frame.

Consider a mechanical system described (with respect to Σ) by a Lagrangian $\mathcal{L} \in C^\infty(J^1\tau)$ and dissipative forces Q_i, subject to a kinetic constraint C locally defined by equation (18). Let $Z \in K$ be a vector field on C projecting onto ΣZ^0. Thus Z takes the local form

$$Z = Z^0 \frac{\partial}{\partial t} + Z^0 \Sigma^i \frac{\partial}{\partial q^i} + Z^a \frac{\partial}{\partial z^a}$$

for some coefficients Z^a, with the Σ^i satisfying the equation

$$\frac{\partial \phi^\mu}{\partial \dot{q}^i}(\Sigma^i - \psi^i) = 0. \tag{61}$$

Then equation (50) is satisfied with $f = 0$, whereas (52) becomes

$$i^*(L_{\dot{\Sigma}}\mathcal{L})Z^0 + \left[\widehat{\mathcal{L}} + \pi_i(\Sigma^i - \psi^i)\right]\dot{Z}^0 + Q_i(\Sigma^i - \psi^i)Z^0 = 0$$

where $\dot{\Sigma} \in \mathcal{D}(J^1\tau)$ is the natural lifting on $J^1\tau$ of Σ. For this equation to hold it is enough that the following relations are satisfied:

$$L_{\dot{\Sigma}}\mathcal{L} = 0 \tag{62}$$

$$\left[\widehat{\mathcal{L}} + \pi_i(\Sigma^i - \psi^i)\right]\dot{Z}^0 + Q_i(\Sigma^i - \psi^i)Z^0 = 0. \tag{63}$$

The first condition implies that the vector field Σ is a symmetry of the Lagrangian. Further insight into (63) can be obtained by considering the following two cases.

(a) Take $Z^0 = 1$ and gyroscopic forces Q_i satisfying the condition $Q_i(\dot{q}^i - \Sigma^i) = 0$, e.g. $Q_i = \gamma_{ij}(\dot{q}^j - \Sigma^j)$ with $\gamma_{ij} = -\gamma_{ji}$; then condition (63) holds. From (53) we find

$$Z^a = \Lambda^a_i \left(\dot{\Sigma}^i - \frac{\partial \psi^i}{\partial t} - \frac{\partial \psi^i}{\partial q^j}\Sigma^j \right)$$

and (54) produces the first integral

$$F = \widehat{\mathcal{L}} + \pi_j(\Sigma^j - \psi^j).$$

(b) Let the Lagrangian take the form

$$\mathcal{L} = \tfrac{1}{2} g_{ij}(\dot{q}^i - \Sigma^i)(\dot{q}^j - \Sigma^j).$$

Then the term $\widehat{\mathcal{L}} + \pi_i(\Sigma^i - \psi^i)$ coincides with $-\widehat{\mathcal{L}}$, so that (63) holds if the dissipative forces are of the form

$$Q_i = -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\frac{\dot{Z}^0}{Z^0}.$$

Note that these are dissipative forces of the Rayleigh type with the dissipative function $-\mathcal{L}\dot{Z}^0/2Z^0$ [1]. Solving equation (53) we find

$$Z^a = \Lambda^a_i \left[\left(\dot{\Sigma}^i - \frac{\partial \psi^i}{\partial t} - \frac{\partial \psi^i}{\partial q^j}\Sigma^j \right) Z^0 + \frac{1}{2}(\Sigma^i - \psi^i)\dot{Z}^0 \right].$$

Moreover, to each vector field Z constructed in this way there corresponds a first integral $F = \widehat{\mathcal{L}}Z^0$.

If we work in a coordinate chart in which the coefficients Σ^i vanish identically, then conditions (61) and (62) take the simple form $\psi^i \partial\phi^\mu/\partial\dot{q}^i = 0$ (homogeneity of the constraint functions with respect to the $\dot{q}^i$) and $\partial\mathcal{L}/\partial t = 0$ (time independence of the Lagrangian).

Example 2. Consider a planar two-particle system whose Lagrangian is

$$\mathcal{L} = T - V = \tfrac{1}{2}\sum_{j=1}^{4}\dot{x}_j^2 - \frac{1}{\rho^2} \qquad \rho^2 = \sum_{j=1}^{4} x_j^2.$$

The system is subject to a kinetic constraint

$$\phi(t, x_j, l_1, l_2) = 0 \tag{64}$$

where $l_1 = x_2\dot{x}_1 - x_1\dot{x}_2$, $l_2 = x_4\dot{x}_3 - x_3\dot{x}_4$ and ϕ is a homogeneous function with respect to l_1 and l_2 (see [9] for an interesting application of angular momentum constraints to the reduction of rigid-body dynamics).

Here we take $E \cong \mathbb{R} \times \mathbb{R}^4$, with coordinates (t, x_j). It is easy to verify that $Z^0 = t$, $Z^j = x_j/2$, $f = 0$ is a solution of equations (46), (50) and (52). Then (54) produces the first integral

$$F = (T + V)t - \tfrac{1}{2}\sum_{j=1}^{4}\psi^j x_j \tag{65}$$

and from (53) we find

$$Z^a = \sum_{j=1}^{4}\Lambda^a_j\left(-\frac{\psi^j}{2} - \frac{\partial\psi^j}{\partial t} - \sum_{k=1}^{4}\frac{\partial\psi^j}{\partial x_k}\frac{x_k}{2}\right). \tag{66}$$

Now let ρ_1, ϑ_1 and ρ_2, ϑ_2 be polar coordinates corresponding to the Cartesian coordinates of the particles. In terms of these coordinates, we can write the constraint equation in the form

$$\phi(t, x_j, -\rho_1^2\dot{\vartheta}_1, -\rho_2^2\dot{\vartheta}_2) = 0.$$

By means of the rank condition (19), we can solve this equation locally with respect to one of the angular velocities $\dot{\vartheta}_1$ and $\dot{\vartheta}_2$. If, for example, $\dot{\vartheta}_2 = \varphi(t, x_j, \rho_1^2\dot{\vartheta}_1)/\rho_2^2$, we let $\dot{\rho}_1$, $\dot{\rho}_2$, $\dot{\vartheta}_1$ play the role of the z^a. A straightforward computation leads then to the following expressions for the functions ψ^j:

$$\psi^1 = \dot{\rho}_1\frac{x_1}{\rho_1} - x_2\dot{\vartheta}_1 \qquad \psi^2 = \dot{\rho}_1\frac{x_2}{\rho_1} + x_1\dot{\vartheta}_1$$
$$\psi^3 = \dot{\rho}_2\frac{x_3}{\rho_2} - \varphi\frac{x_4}{\rho_2^2} \qquad \psi^4 = \dot{\rho}_2\frac{x_4}{\rho_2} + \varphi\frac{x_3}{\rho_2^2}$$

and the first integral (65) takes the form

$$F = (T + V)t - \frac{1}{4}\frac{\mathrm{d}\rho^2}{\mathrm{d}t}.$$

If, in particular, the function ϕ does not depend on t and x_j, then (66) yields $Z^a = -\Lambda^a_i\psi^i$ and a straightforward computation yields $Z^1 = -\dot{\rho}_1$, $Z^2 = -\dot{\rho}_2$ and $Z^3 = -\dot{\vartheta}_1$.

Example 3. In this example we consider the tippe top. We make the following assumptions. The top is a sphere whose centre of mass does not coincide with its geometrical centre. During its motion the body remains in contact with a fixed horizontal plane. The friction at the contact point prevents the top from sliding.

Introduce the Euler angles ψ, θ, ϕ using the principal axis body frame relative to an inertial reference frame (in the notation used here, ϕ is the angle of rotation about the symmetry axis of the top). These angles together with two horizontal coordinates (x, y) of the centre of mass are coordinates in the configuration space $\mathbb{R}^2 \times SO(3)$ of the tippe top. Here $E \cong \mathbb{R} \times \mathbb{R}^2 \times SO(3)$.

The Lagrangian of the top is computed to be

$$\mathcal{L} = \tfrac{1}{2}(A\sin^2\theta + C\cos^2\theta)\dot{\psi}^2 + \tfrac{1}{2}(A + ma^2\sin^2\theta)\dot{\theta}^2 + \tfrac{1}{2}C\dot{\phi}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + C\cos\theta\dot{\psi}\dot{\phi} - mga\cos\theta \tag{67}$$

where A and C are the principal moments of inertia of the body, m is its total mass and a is the distance of the centre of mass from the centre of the sphere. The constraints are

$$\dot{x} + \sin\psi(-r + a\cos\theta)\dot{\theta} + \sin\theta\cos\psi(r\dot{\phi} + a\dot{\psi}) = 0 \tag{68}$$

$$\dot{y} - \cos\psi(-r + a\cos\theta)\dot{\theta} + \sin\theta\sin\psi(r\dot{\phi} + a\dot{\psi}) = 0. \tag{69}$$

Note that the symmetry group of the Lagrangian and the constraints is the subgroup $SE(2) \times SO(2)$ of $\mathbb{R}^2 \times SO(3)$ generated by translations parallel to the horizontal plane, by rotations about the vertical axis through the origin of the inertial frame and by rotations around the symmetry axis of the top. The action of an element $(a, b, \alpha, \beta) \in SE(2) \times SO(2)$ is given by

$$(x, y, \psi, \theta, \phi) \mapsto (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \psi + \alpha, \theta, \phi + \beta). \tag{70}$$

For any element ζ in the Lie algebra of $SE(2) \times SO(2)$ we denote by Y_ζ the corresponding infinitesimal generator. Setting $\zeta = (a', b', \alpha', \beta')$ we obtain the vector field on E given by

$$Y_\zeta = (-y\alpha' + a')\frac{\partial}{\partial x} + (x\alpha' + b')\frac{\partial}{\partial y} + \alpha'\frac{\partial}{\partial\psi} + \beta'\frac{\partial}{\partial\phi}. \tag{71}$$

Let $\dot{\psi}, \dot{\theta}$ and $\dot{\phi}$ play the role of z^a in the general theory. We seek solutions to equations (46), (50) and (52) which are vector fields tangent to the orbits of the symmetry group. Writing out (46) in terms of the infinitesimal generators, we find the equations

$$-y\alpha' + a' + \alpha' a\sin\theta\cos\psi + \beta' r\sin\theta\cos\psi = 0 \tag{72}$$

$$x\alpha' + b' + \alpha' a\sin\theta\sin\psi + \beta' r\sin\theta\sin\psi = 0. \tag{73}$$

Using these equations, we can express the vector fields we are looking for as pointwise linear combinations of the above infinitesimal generators (cf [3]). Clearly, equation (50) is satisfied with $f = 0$ since $Z^0 = 0$ and Z^i do not depend on the coordinates z^a. Equation (52) reduces to

$$\frac{\partial\mathcal{L}}{\partial q^i}Z^i + \pi_i\frac{\partial Z^i}{\partial q^j}\psi^j = 0 \tag{74}$$

or, more concisely, $i^*(L_{\dot{Y}_\zeta}\mathcal{L}) = 0$. Although $SE(2) \times SO(2)$ is a symmetry group of the Lagrangian, equation (74) need not be satisfied in general since we are working with pointwise linear combinations of the infinitesimal generators. However, the infinitesimal generator

$$r\frac{\partial}{\partial\psi} - a\frac{\partial}{\partial\phi}$$

of the Lie algebra element $a' = ry, b' = -rx, \alpha' = r, \beta' = -a$ satisfies equation (74). Note that it rotates the top while fixing the centre of mass. The corresponding first integral, which is known as Jellet's integral (see [3] and references therein), is given by equation (54):

$$F = aC(\dot{\psi}\cos\theta + \dot{\phi}) - r\left[(A\sin^2\theta + C\cos^2\theta)\dot{\psi} + C\cos\theta\dot{\phi}\right].$$

This quantity is nothing but the projection of the angular momentum onto the vector $\vec{GT}$, where G is the centre of mass and T is the instantaneous point of contact of the top.

Example 4. A simple example that illustrates the use of equations (47) and (48) is the following example of a non-holonomically constrained free particle [26]. Consider a particle with the Lagrangian $\mathcal{L} = 1/2(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ and the non-holonomic constraint $\dot{z} = y\dot{x}$. Here $E \cong \mathbb{R}\times\mathbb{R}^3$, with coordinates (t, x, y, z). Let $\dot{x}$, $\dot{y}$ play the role of the z^a in our discussion of the general theory. Then (16) takes the form

$$\psi_1 = \dot{x} \qquad \psi_2 = \dot{y} \qquad \psi_3 = y\dot{x}.$$

According to (36), the SODE field on C here becomes

$$\widehat{\xi} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + y\dot{x}\frac{\partial}{\partial z} - \frac{y\dot{x}\dot{y}}{1+y^2}\frac{\partial}{\partial \dot{x}}. \tag{75}$$

As we know from example 1, a first integral of (75) is provided by the energy

$$\widehat{\mathcal{L}} = \tfrac{1}{2}\left[(1+y^2)\dot{x}^2 + \dot{y}^2\right].$$

From (47) and (48) we find all vector fields generating the energy integral, which are given by

$$Z = Z^0\frac{\partial}{\partial t} + (Z^0 - 1)\left[\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + y\dot{x}\frac{\partial}{\partial z} - \frac{y\dot{x}\dot{y}}{1+y^2}\frac{\partial}{\partial \dot{x}}\right].$$

The differential equations corresponding to the vector field (75) are given by

$$\ddot{x} + \frac{y\dot{x}\dot{y}}{1+y^2} = 0 \qquad \ddot{y} = 0 \qquad \dot{z} = y\dot{x}.$$

Note that the first equation is equivalent to $\mathrm{d}/\mathrm{d}t\left[\dot{x}(1+y^2)^{1/2}\right] = 0$, so that the function

$$F = \dot{x}(1+y^2)^{1/2} \tag{76}$$

is a first integral of $\widehat{\xi}$. This integral was used in the Bates–Sniatycki reduction [2]. According to equations (47) and (48), the (unique) vertical vector field corresponding to the invariant (76) is given by

$$Z = -\frac{1}{(1+y^2)^{1/2}}\frac{\partial}{\partial x} - \frac{y}{(1+y^2)^{1/2}}\frac{\partial}{\partial z}.$$

Also note that in this case $L_Z\widehat{\omega}_{\mathcal{L}} = 0$, hence $F = Z\rfloor\widehat{\omega}_{\mathcal{L}}$.

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